

**PERTURBATIVE QUANTUM GRAVITY AND NEWTON'S LAW
ON A FLAT ROBERTSON-WALKER BACKGROUND****J. Iliopoulos***Laboratoire de Physique Théorique, Ecole Normale Supérieure
24, rue Lhomond, F-75231 Paris Cedex 05, FRANCE
ilio@physique.ens.fr***T. N. Tomaras , N. C. Tsamis***Department of Physics, University of Crete
and Research Center of Crete
P. O. Box 2208, GR-71003 Heraklion, GREECE
tomaras@physics.ucl.ac.uk , tsamis@physics.ucl.ac.uk***R. P. Woodard***Department of Physics, University of Florida
Gainesville, FL 32611, USA
woodard@phys.ufl.edu***ABSTRACT**

We derive the Feynman rules for the graviton in the presence of a flat Robertson-Walker background and give explicit expressions for the propagator in the physically interesting cases of inflation, radiation domination, and matter domination. The aforementioned background is generated by a scalar field source which should be taken to be dynamical. As an elementary application, we compute the corrections to the Newtonian gravitational force in the present matter dominated era and conclude – as expected – that they are negligible except for the largest scales.

1. Introduction

Of the non-trivial spacetime backgrounds, those of particular interest in cosmology are spatially homogeneous and isotropic. This is a direct consequence of the cosmological principle which states that there is no preferred position in the universe. It can be shown that the most general background satisfying the above requirements is the Robertson-Walker one [1]:

$$d\hat{s}^2 = -dt^2 + a^2(t) \left\{ (1 - kr^2)^{-1} dr^2 + r^2 d\Omega_2^2 \right\} . \quad (1.1)$$

It is characterized by a function $a(t)$ and a constant k which can be chosen to take on the values $+1$, 0 , or -1 . The function $a(t)$ – known as the scale factor – is a measure of the “radius” of the universe, while the constant k determines the spatial curvature.

By far the most natural explanation of the observed homogeneity and isotropy in the present universe is the assumption of an inflationary era in the evolution of the universe. During that era, the physical distance between observers at rest on fixed spatial coordinates increases superluminally. Accordingly, the observed universe was once so small that causal processes could have established an initial thermal equilibrium across it. This accounts for the fact that the cosmic microwave background radiation from different regions of the sky is seen to be in thermal equilibrium to within one part in 10^5 [2]. Since any decent amount of inflation * redshifts the spatial curvature to insignificance, we can restrict our study to the $k = 0$ geometries:

$$d\hat{s}^2 = -dt^2 + a^2(t) d\vec{x} \cdot d\vec{x} . \quad (1.2)$$

Such geometries do not represent solutions of the Einstein equations in vacuum except in the case of flat space where the scale factor is equal to one. In all other cases a non-trivial stress tensor must be present. Homogeneity and isotropy constrain the stress tensor to have only diagonal elements consisting of the density $\rho(t)$ and pressure $p(t)$:

* By decent we mean at least the 60 e-foldings of inflation needed to explain the observed degree of isotropy.

$T_{\mu\nu} = [\rho(t) p(t) p(t) p(t)]$. This form can arise either from a non-zero temperature matter theory or a non-trivial scalar field in zero temperature quantum field theory. To preserve the isotropy of space, the matter fields must be scalar since the Lorentz transformation properties of all other fields – like fermions or gauge bosons – exhibit preferred directions. The determination of the scalar field action which supports an arbitrary spatially flat Robertson-Walker background is given in Section 2.

It is best to set-up the theory on an arbitrary spatial manifold of finite extent – for instance, $T^3 \times \mathfrak{R}$ – in which case $\Delta x^i \leq R$. We shall find it very convenient to utilize the conformal set of coordinates:

$$d\hat{s}^2 = -dt^2 + a^2(t) d\vec{x} \cdot d\vec{x} = \Omega^2(\eta) \left(-d\eta^2 + d\vec{x} \cdot d\vec{x} \right) . \quad (1.3a)$$

The relation between the two coordinate systems is given by:

$$dt = \Omega(\eta) d\eta \quad ; \quad a(t) = \Omega(\eta) , \quad (1.3b)$$

so that a generic power law in co-moving coordinates:

$$a(t) = \left(\frac{t}{t_0} \right)^s , \quad (1.4a)$$

takes the following form in conformal coordinates:

$$\Omega(\eta) = \left(\frac{\eta}{\eta_0} \right)^{\frac{s}{1-s}} , \quad (1.4b)$$

with $\eta_0 = t_0 (1-s)^{-1}$. The physically most distinguishable cases consist of the *inflating* universe for $s = +\infty$, the *radiation dominated* universe for $s = \frac{1}{2}$, the *matter dominated* universe for $s = \frac{2}{3}$, and the *flat* universe for $s = 0$. We shall also assume throughout that $\Delta x \ll R$ and $\Delta \eta \ll R$ since then the integral approximation to mode sums which appear because we work on T^3 is excellent. *

* In conformal coordinates light reaches the distance $\Delta x = R$ in time $\Delta \eta = R$.

In order to study gravitational effects not associated with the smallest of scales in the class of backgrounds (1.2), we must develop the appropriate perturbative tools – this is accomplished in Section 3. As an elementary application of the perturbative results, we calculate in Section 4 the response of the gravitational field due to a point source in a matter dominated universe. The resulting corrections to the Newtonian long-range gravitational force are found to be negligible as expected. Our conclusions comprise Section 5.

2. Determination of the source background

The dynamical system under consideration has a background action which consists of two parts:

$$\widehat{\mathcal{S}} = \widehat{\mathcal{S}}_{\text{g}} + \widehat{\mathcal{S}}_{\text{m}} \quad . \quad (2.1)$$

The gravitational action $\widehat{\mathcal{S}}_{\text{g}}$ is obtained from the Einstein Lagrangian:

$$\widehat{\mathcal{L}}_{\text{g}} = \frac{1}{16\pi G} R \sqrt{-g} \quad , \quad (2.2a)$$

while the matter action $\widehat{\mathcal{S}}_{\text{m}}$ is based on a generic scalar field Lagrangian:

$$\widehat{\mathcal{L}}_{\text{m}} = -\frac{1}{2} \partial_{\mu}\varphi \partial_{\nu}\varphi g^{\mu\nu} \sqrt{-g} - V(\varphi) \sqrt{-g} \quad . \quad (2.2b)$$

We obtain the following equations of motion for the gravitational field:

$$G^{\mu\nu} = (8\pi G) T^{\mu\nu} \quad , \quad (2.3a)$$

and for the scalar field:

$$\partial_{\mu} \left(\sqrt{-g} g^{\mu\nu} \partial_{\nu} \varphi \right) = \frac{\partial V(\varphi)}{\partial \varphi} \sqrt{-g} \quad . \quad (2.3b)$$

The Einstein tensor $G^{\mu\nu}$ is determined by the geometry (1.2): *

$$G^{00} = 3 \left(\frac{\dot{a}}{a} \right)^2 \quad ; \quad G^{ij} = -g^{ij} \left(2 \frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} \right) \quad , \quad (2.4)$$

* Our metric has spacelike signature and $R^{\mu}_{\nu\rho\sigma} = \Gamma^{\mu}_{\sigma\nu,\rho} + \dots$ Greek indices run from 0 to 3, while Latin indices run from 1 to 3. Dots as superscripts denote differentiation with respect to the co-moving time t . We are also using units where $c = \hbar = 1$.

and the covariantly conserved stress tensor $T^{\mu\nu}$ by the matter sector of the theory:

$$T^{\mu\nu} \equiv \frac{2}{\sqrt{-g}} \frac{\delta \mathcal{S}_m}{\delta g_{\mu\nu}} = \left(g^{\mu\alpha} g^{\nu\beta} - \frac{1}{2} g^{\mu\nu} g^{\alpha\beta} \right) \partial_\alpha \varphi \partial_\beta \varphi - g^{\mu\nu} V(\varphi) , \quad (2.5a)$$

$$T^{\mu\nu}{}_{;\nu} = 0 . \quad (2.5b)$$

The resulting density $\rho(t)$ and pressure $p(t)$ take the form:

$$T^{00} \equiv \rho = \frac{1}{2} \dot{\hat{\phi}}^2 + V(\hat{\phi}) , \quad (2.6a)$$

$$T^{ij} \equiv p g^{ij} = g^{ij} \left(\frac{1}{2} \dot{\hat{\phi}}^2 - V(\hat{\phi}) \right) . \quad (2.6b)$$

In order not to disturb the spatial homogeneity and isotropy of our background geometries, we have taken the scalar field to be a function of time only: $\varphi = \hat{\phi}(t)$.

Using (2.4) and (2.6), the gravitational equations (2.3a) become:

$$\dot{\hat{\phi}}^2 = \frac{1}{4\pi G} \left(-\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} \right) , \quad (2.7a)$$

$$V(\hat{\phi}) = \frac{1}{8\pi G} \left(\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} \right) . \quad (2.7b)$$

By satisfying equations (2.7), we determine the source parameters $\hat{\phi}(t)$ and $V(\hat{\phi})$ – or, equivalently, $\rho(t)$ and $p(t)$ – which support the flat Robertson-Walker background (1.2).

In conformal coordinates the above equations take the form: *

$$\dot{\hat{\phi}}^2 = \frac{1}{4\pi G} \left(-\frac{\Omega''}{\Omega^3} + \frac{2\Omega'^2}{\Omega^4} \right) , \quad (2.8a)$$

$$V(\hat{\phi}) = \frac{1}{8\pi G} \left(\frac{\Omega''}{\Omega^3} + \frac{\Omega'^2}{\Omega^4} \right) , \quad (2.8b)$$

where the primes over Ω indicate differentiation with respect to the conformal time η . It remains to be shown that the above choice for the source parameters is consistent with the scalar equation of motion (2.3b):

$$\ddot{\hat{\phi}} + \frac{3\dot{a}}{a} \dot{\hat{\phi}} + \frac{\partial V(\hat{\phi})}{\partial \hat{\phi}} = 0 . \quad (2.9)$$

* Notice that in the case of de Sitter spacetime, $s \rightarrow +\infty$, equations (2.7-8) have the proper limit: $\dot{\hat{\phi}}^2 \rightarrow 0$ and $V(\hat{\phi}) \rightarrow (8\pi G) \Lambda^{-1}$, where $\Lambda = 3\eta_0^{-2}$.

This is a direct consequence of the conservation equation (2.5b):

$$\dot{\rho} = -\frac{3\dot{a}}{a} (\rho + p) . \quad (2.10)$$

By substituting (2.6) in (2.10) we obtain:

$$\frac{d}{dt} \left(\frac{1}{2} \dot{\hat{\phi}}^2 + V(\hat{\phi}) \right) = -\frac{3\dot{a}}{a} \dot{\hat{\phi}}^2 , \quad (2.11)$$

which is identical to (2.9).

3. The Feynman rules

3.1 The Lagrangian.

The quantum fields are the graviton $h_{\mu\nu}(x)$ and the scalar $\phi(x)$:

$$g_{\mu\nu} \equiv \hat{g}_{\mu\nu} + \kappa h_{\mu\nu} , \quad (3.1a)$$

$$\varphi \equiv \hat{\phi} + \phi . \quad (3.1b)$$

It turns out that – just like the case of de Sitter spacetime [3] – it is most convenient to organize perturbation theory in terms of the “pseudo-graviton” field, $\psi_{\mu\nu}(x)$, obtained by conformally re-scaling the metric:

$$g_{\mu\nu} \equiv \Omega^2 \tilde{g}_{\mu\nu} \equiv \Omega^2 \left(\eta_{\mu\nu} + \kappa \psi_{\mu\nu} \right) . \quad (3.2)$$

As usual, pseudo-graviton indices are raised and lowered with the Lorentz metric and $\kappa^2 \equiv 16\pi G$. The total Lagrangian is:

$$\mathcal{L} = \frac{1}{\kappa^2} R \sqrt{-g} - \frac{1}{2} \partial_\mu \varphi \partial_\nu \varphi g^{\mu\nu} \sqrt{-g} - V(\varphi) \sqrt{-g} . \quad (3.3)$$

By substituting (3.1b) in (3.3) we get:

$$\begin{aligned} \mathcal{L} = & \frac{1}{\kappa^2} R \sqrt{-g} - \frac{1}{2} \hat{\phi}'^2 g^{00} \sqrt{-g} - \partial_\mu \hat{\phi} \partial_\nu \phi g^{\mu\nu} \sqrt{-g} - \frac{1}{2} \partial_\mu \phi \partial_\nu \phi g^{\mu\nu} \sqrt{-g} \\ & - \sum_{n=0}^{+\infty} \frac{1}{n!} \frac{\partial^n V}{\partial \varphi^n}(\hat{\phi}) \phi^n \sqrt{-g} , \end{aligned} \quad (3.4)$$

and we can proceed to organize \mathcal{L} according to the number of fields ϕ present.

The pure metric part of \mathcal{L} :

$$\mathcal{L}_{(0)} = \frac{1}{\kappa^2} R \sqrt{-g} - \frac{1}{2} \hat{\phi}'^2 g^{00} \sqrt{-g} - V(\hat{\phi}) \sqrt{-g} , \quad (3.5)$$

in terms of the re-scaled metric is:

$$\mathcal{L}_{(0)} = \frac{1}{\kappa^2} \left\{ \Omega^2 \tilde{R} + 2(-1 + \tilde{g}^{00}) (\Omega \Omega'' + \Omega'^2) \right\} \sqrt{-\tilde{g}} , \quad (3.6)$$

where we used (2.8) and ignored surface terms. After performing many partial integrations, (3.6) can be cast in a form identical to that of de Sitter spacetime [3]:

$$\begin{aligned} \mathcal{L}_{(0)} = \sqrt{-\tilde{g}} \tilde{g}^{\alpha\beta} \tilde{g}^{\rho\sigma} \tilde{g}^{\mu\nu} & \left[\frac{1}{2} \psi_{\alpha\rho,\mu} \psi_{\nu\sigma,\beta} - \frac{1}{2} \psi_{\alpha\beta,\rho} \psi_{\sigma\mu,\nu} + \frac{1}{4} \psi_{\alpha\beta,\rho} \psi_{\mu\nu,\sigma} - \frac{1}{4} \psi_{\alpha\rho,\mu} \psi_{\beta\sigma,\nu} \right] \Omega^2 \\ & - \frac{1}{2} \sqrt{-\tilde{g}} \tilde{g}^{\rho\sigma} \tilde{g}^{\mu\nu} \psi_{\rho\sigma,\mu} \psi_{\nu}{}^{\alpha} (\Omega^2)_{,\alpha} . \end{aligned} \quad (3.7)$$

All the self-interactions of the graviton in the presence of an arbitrary scale factor $\Omega(\eta)$ can be obtained by expanding expression (3.7). The quadratic part of $\mathcal{L}_{(0)}$ is:

$$\begin{aligned} \mathcal{L}_{(0)}^{(2)} = & \left[\frac{1}{2} \psi^{\alpha\rho,\mu} \psi_{\mu\rho,\alpha} - \frac{1}{2} \psi_{,\alpha} \psi^{\alpha\rho}{}_{,\rho} + \frac{1}{4} \psi_{,\alpha} \psi_{,\alpha} - \frac{1}{4} \psi^{\alpha\rho,\mu} \psi_{\alpha\rho,\mu} \right] \Omega^2 \\ & + \psi^{,\mu} \psi_{\mu 0} \Omega \Omega' , \end{aligned} \quad (3.8)$$

where $\psi \equiv \psi^\mu_\mu$.

The part of \mathcal{L} linear in ϕ is simpler:

$$\mathcal{L}_{(1)} = -\hat{\phi}' \partial_\nu \phi g^{0\nu} \sqrt{-g} - \frac{\partial V}{\partial \phi}(\hat{\phi}) \phi \sqrt{-g} , \quad (3.9a)$$

$$= \frac{1}{\kappa} \left[-\xi \partial_\mu \phi \tilde{g}^{\mu 0} + \xi' \phi \right] \sqrt{-\tilde{g}} . \quad (3.9b)$$

In the last step we used the scalar background equation of motion (2.9) to derive the identity:

$$\frac{\partial V}{\partial \phi}(\hat{\phi}) = -\frac{1}{\kappa \Omega^4} \xi' , \quad (3.10a)$$

where we define $\xi(\eta)$ as:

$$\xi \equiv \kappa \Omega^2 \widehat{\phi}' . \quad (3.10b)$$

The following quadratic part emerges:

$$\mathcal{L}_{(1)}^{(2)} = \xi \partial_\mu \phi \psi^{\mu 0} + \frac{1}{2} (\xi \phi)' \psi . \quad (3.11)$$

We shall also need the part of \mathcal{L} which is quadratic in ϕ :

$$\mathcal{L}_{(2)} = -\frac{1}{2} \partial_\mu \phi \partial_\nu \phi g^{\mu\nu} \sqrt{-g} - \frac{1}{2} \frac{\partial^2 V}{\partial \varphi^2}(\widehat{\phi}) \phi^2 \sqrt{-g} , \quad (3.12a)$$

$$= -\frac{1}{2} \Omega^2 \partial_\mu \phi \partial_\nu \phi \widetilde{g}^{\mu\nu} \sqrt{-\widetilde{g}} + \frac{1}{2} \Omega^6 \xi^{-1} (\Omega^{-4} \xi')' \phi^2 \sqrt{-\widetilde{g}} , \quad (3.12b)$$

and the associated piece which is quadratic in the quantum fields:

$$\mathcal{L}_{(2)}^{(2)} = -\frac{1}{2} \Omega^2 \partial_\mu \phi \partial^\mu \phi + \frac{1}{2} \Omega^6 \xi^{-1} (\Omega^{-4} \xi')' \phi^2 . \quad (3.13)$$

In order to properly quantize the theory and calculate the various propagators, we must fix the gauge. We shall accomplish this by adding the following term to the Lagrangian:

$$\mathcal{L}_{\text{GF}} = -\frac{1}{2} \eta_{\mu\nu} F^\mu F^\nu , \quad (3.14a)$$

where:

$$F_\mu = \Omega \psi_\mu{}^\nu{}_{,\nu} - \frac{1}{2} \Omega \psi_{,\mu} - 2\Omega' \psi_{\mu 0} + \Omega^{-1} \eta_{\mu 0} \xi \phi . \quad (3.14b)$$

The resulting gauge-fixed quadratic Lagrangian is:

$$\begin{aligned} \mathcal{L}_{\text{GF}}^{(2)} &= \mathcal{L}_{(0)}^{(2)} + \mathcal{L}_{(1)}^{(2)} + \mathcal{L}_{(2)}^{(2)} + \mathcal{L}_{\text{GF}} \\ &= -\frac{1}{8} \psi \Omega \left[\partial^2 + \frac{\Omega''}{\Omega} \right] \Omega \psi + \frac{1}{4} \psi^{\mu\nu} \Omega \left[\partial^2 + \frac{\Omega''}{\Omega} \right] \Omega \psi_{\mu\nu} \\ &\quad + \psi^{\mu 0} \Omega \left[-\frac{\Omega''}{\Omega} + \frac{\Omega'^2}{\Omega^2} \right] \Omega \psi_{\mu 0} + \psi_{00} \Omega \left[-\Omega^{-2} \xi' + 2 \frac{\Omega'}{\Omega^3} \xi \right] \Omega \phi \\ &\quad + \frac{1}{2} \phi \Omega \left[\partial^2 + \frac{\Omega''}{\Omega} + \Omega^{-4} \xi^2 + \Omega^4 \xi^{-1} (\Omega^{-4} \xi')' \right] \Omega \phi . \end{aligned} \quad (3.15)$$

In terms of a kinetic operator $D_{\mu\nu}^{\rho\sigma}$ we get:

$$\begin{aligned} \mathcal{L}_{\text{GF}}^{(2)} \equiv & + \frac{1}{2} \psi^{\mu\nu} D_{\mu\nu}^{\rho\sigma} \psi_{\rho\sigma} + \psi_{00} \Omega \left[-\Omega^{-2} \xi' + 2 \frac{\Omega'}{\Omega^3} \xi \right] \Omega \phi \\ & + \frac{1}{2} \phi \Omega \left[\partial^2 + \frac{\Omega''}{\Omega} + \Omega^{-4} \xi^2 + \Omega^4 \xi^{-1} (\Omega^{-4} \xi')' \right] \Omega \phi . \end{aligned} \quad (3.16)$$

where we have:

$$\begin{aligned} D_{\mu\nu}^{\rho\sigma} = & \left[\frac{1}{2} \bar{\delta}_{\mu}^{(\rho} \bar{\delta}_{\nu}^{\sigma)} - \frac{1}{4} \eta_{\mu\nu} \eta^{\rho\sigma} - \frac{1}{2} \delta_{\mu}^0 \delta_{\nu}^0 \delta_0^{\rho} \delta_0^{\sigma} \right] D_A \\ & + \delta_{(\mu}^0 \bar{\delta}_{\nu)}^{(\rho} \delta_0^{\sigma)} D_B + \delta_{\mu}^0 \delta_{\nu}^0 \delta_0^{\rho} \delta_0^{\sigma} D_B . \end{aligned} \quad (3.17)$$

Parenthesized indices are symmetrized and a bar above a Lorentz metric or a Kronecker delta symbol means that the zero (i.e., η) component is projected out:

$$\bar{\eta}_{\mu\nu} \equiv \eta_{\mu\nu} + \delta_{\mu}^0 \delta_{\nu}^0 \quad ; \quad \bar{\delta}_{\mu}^{\nu} \equiv \delta_{\mu}^{\nu} - \delta_{\mu}^0 \delta_0^{\nu} . \quad (3.18)$$

The quadratic operators defined in (3.17) are given by:

$$D_A \equiv \Omega \left[\partial^2 + \frac{\Omega''}{\Omega} \right] \Omega , \quad (3.19a)$$

$$D_B \equiv \Omega \left[\partial^2 - \frac{\Omega''}{\Omega} + 2 \frac{\Omega'^2}{\Omega^2} \right] \Omega . \quad (3.19b)$$

Notice that D_A is the kinetic operator for a massless, minimally coupled scalar in the presence of the gravitational background (1.3).

3.2 The Diagonal Variables.

The quadratic Lagrangian (3.16) should be brought into diagonal form. There is mixing in the first term of (3.16) between ψ_{ii} and ψ_{00} :

$$\begin{aligned} \frac{1}{2} \psi^{\mu\nu} D_{\mu\nu}^{\rho\sigma} \psi_{\rho\sigma} = & \frac{1}{2} \left\{ \frac{1}{2} \psi_{ij} D_A \psi_{ij} - \frac{1}{4} (\psi_{ii} - \psi_{00}) D_A (\psi_{jj} - \psi_{00}) - \frac{1}{2} \psi_{00} D_A \psi_{00} \right. \\ & \left. - \psi_{0i} D_B \psi_{0i} + \psi_{00} D_B \psi_{00} \right\} , \end{aligned} \quad (3.20)$$

which is removed by transforming to the field variable $\zeta_{\mu\nu}$:

$$\begin{aligned} \frac{1}{2} \psi^{\mu\nu} D_{\mu\nu}^{\rho\sigma} \psi_{\rho\sigma} = & \frac{1}{2} \left\{ \frac{1}{2} \zeta_{ij} D_A \zeta_{ij} - \frac{1}{4} \zeta_{ii} D_A \zeta_{jj} - \zeta_{0i} D_B \zeta_{0i} + \zeta_{00} D_B \zeta_{00} \right\} \\ = & \frac{1}{2} \left\{ \zeta_{ij} D_A \zeta_{rs} \left[\frac{1}{2} \delta_{i(r} \delta_{s)j} - \frac{1}{4} \delta_{ij} \delta_{rs} \right] + \zeta_{0i} D_B \zeta_{0r} \left[-\delta_{ir} \right] \right. \\ & \left. + \zeta_{00} D_B \zeta_{00} \right\} , \end{aligned} \quad (3.21)$$

where we defined:

$$\psi_{ij} \equiv \zeta_{ij} + \delta_{ij} \zeta_{00} \quad \psi_{0i} \equiv \zeta_{0i} \quad \psi_{00} \equiv \zeta_{00} \quad . \quad (3.22)$$

There is also mixing between ζ_{00} and ϕ . The part of the quadratic Lagrangian involved is:

$$\begin{aligned} \mathcal{L}_{\text{mixing}} = & \frac{1}{2} \zeta_{00} D_B \zeta_{00} + \zeta_{00} \Omega \left[-\Omega^{-2} \xi' + 2 \frac{\Omega'}{\Omega^3} \xi \right] \Omega \phi \\ & + \frac{1}{2} \phi \Omega \left[\partial^2 + \frac{\Omega''}{\Omega} + \Omega^{-4} \xi^2 + \Omega^4 \xi^{-1} (\Omega^{-4} \xi')' \right] \Omega \phi \quad . \end{aligned} \quad (3.23)$$

We can always diagonalize this system but, for most scale factors $\Omega(\eta)$, the resulting kinetic operators will be non-local. In this case the mode functions obey fourth order differential equations. However, for a class of scale factors which includes the generic power law (1.4), the system can be diagonalized without sacrificing locality or the second order character of the mode equations.* In this case, $\mathcal{L}_{\text{mixing}}$ takes the form:

$$\begin{aligned} \mathcal{L}_{\text{mixing}}^s = & \frac{1}{2} \zeta_{00} \Omega \left[\partial^2 + \frac{s}{(1-s)^2} \frac{1}{\eta^2} \right] \Omega \zeta_{00} + \zeta_{00} \Omega \frac{2\sqrt{s}}{1-s} \frac{1}{\eta^2} \Omega \phi \\ & + \frac{1}{2} \phi \Omega \left[\partial^2 + \frac{2s^2-3s+2}{(1-s)^2} \frac{1}{\eta^2} \right] \Omega \phi \quad . \end{aligned} \quad (3.24)$$

Let us call the diagonal variables χ and v :

$$\zeta_{00} = \cos \theta \chi - \sin \theta v \quad , \quad (3.25a)$$

$$\phi = \sin \theta \chi + \cos \theta v \quad , \quad (3.25b)$$

and demand that the part of $\mathcal{L}_{\text{mixing}}^s$ involving χv vanish:

$$\mathcal{L}_{\chi v}^s = \frac{1}{\eta^2} \chi \Omega \left\{ \sin(2\theta) + \frac{2\sqrt{s}}{1-s} \cos(2\theta) \right\} \Omega v = 0 \quad . \quad (3.26)$$

Of the two available solutions, it is sufficient to consider one of them:

$$\cos^2 \theta = \frac{1}{1+s} \quad ; \quad \sin^2 \theta = \frac{s}{1+s} \quad ; \quad \sin \theta \cos \theta = -\frac{\sqrt{s}}{1+s} \quad , \quad (3.27)$$

* The condition that $\Omega(\eta)$ must obey is: $-\Omega \frac{d}{d\eta} (\Omega^{-2} \Omega') = \frac{c_1}{c_2} \tan^2 (\sqrt{c_1 c_2} \eta + \sqrt{\frac{c_1}{c_2}} \theta)$, where c_1, c_2 are constants and θ is an angle.

since the other merely interchanges the role of χ and v . The result is the following diagonal form:

$$\mathcal{L}_{\text{mixing}}^s = \frac{1}{2}\chi \Omega \left[\partial^2 + \frac{2s^2-s}{(1-s)^2} \frac{1}{\eta^2} \right] \Omega \chi + \frac{1}{2}v \Omega \left[\partial^2 + \frac{2-s}{(1-s)^2} \frac{1}{\eta^2} \right] \Omega v . \quad (3.28)$$

The complete diagonal quadratic Lagrangian is:

$$\begin{aligned} \mathcal{L}_{\text{GF}}^{(2)} = & \frac{1}{2}\zeta_{ij} D_A^s \zeta_{rs} \left[\frac{1}{2}\delta_{i(r} \delta_{s)j} - \frac{1}{4}\delta_{ij} \delta_{rs} \right] + \frac{1}{2}\zeta_{0i} D_B^s \zeta_{0r} \left[-\delta_{ir} \right] \\ & + \frac{1}{2}v D_C^s v + \frac{1}{2}\chi D_A^s \chi , \end{aligned} \quad (3.29)$$

where:

$$D_A^s = \Omega \left[\partial^2 + \frac{2s^2-s}{(1-s)^2} \frac{1}{\eta^2} \right] \Omega , \quad (3.30a)$$

$$D_B^s = \Omega \left[\partial^2 + \frac{s}{(1-s)^2} \frac{1}{\eta^2} \right] \Omega , \quad (3.30b)$$

$$D_C^s = \Omega \left[\partial^2 + \frac{2-s}{(1-s)^2} \frac{1}{\eta^2} \right] \Omega . \quad (3.30c)$$

It becomes transparent that the “A” graviton modes are six and have purely spatial polarizations, the “B” graviton modes are three and have mixed polarizations, and the “C” graviton mode is one with purely temporal polarization. Clearly, the two physical graviton polarizations are transverse traceless “A” modes.

3.3 The Propagators.

Manifest spatial translation invariance allows us to write the general linearized solution as the following superposition:

$$\begin{aligned} \psi_{\mu\nu}(\eta, \vec{x}) = & \sum_{\lambda} \int \frac{d^3k}{(2\pi)^3} \exp[i\vec{k} \cdot \vec{x}] \left\{ \Psi(\eta, k, \lambda) a_{\mu\nu}(\vec{k}, \lambda) + \Psi^*(\eta, k, \lambda) a_{\mu\nu}^\dagger(\vec{k}, \lambda) \right\} , \\ \phi(\eta, \vec{x}) = & \int \frac{d^3k}{(2\pi)^3} \exp[i\vec{k} \cdot \vec{x}] \left\{ \Psi(\eta, k) a(\vec{k}) + \Psi^*(\eta, k) a^\dagger(\vec{k}) \right\} , \end{aligned} \quad (3.31b)$$

where $k = \|\vec{k}\|$ and the index λ spans all available polarizations. The free vacuum of the theory is defined as the normalizable state that obeys: *

$$a_{\mu\nu}(\vec{k}, \lambda) \left| 0 \right\rangle = a(\vec{k}) \left| 0 \right\rangle = 0 . \quad (3.32)$$

* Besides the $a_{\mu\nu}(\vec{k}, \lambda)$, the analogous ghost operators annihilate $|0\rangle$ as well.

The mode functions are canonically normalized:

$$\Psi(\eta, k, \lambda) \Psi^{\star'}(\eta, k, \lambda) - \Psi^{\star}(\eta, k, \lambda) \Psi'(\eta, k, \lambda) = i\Omega^{-2} \quad \forall \lambda \quad , \quad (3.33)$$

and are annihilated upon acting the appropriate quadratic operators. In our case:

$$D_I^s \Psi(\eta, k, I) = 0 \quad , \quad I = A, B, C \quad . \quad (3.34)$$

The pseudo-graviton and scalar propagators are defined by:

$$i \left[{}_{\mu\nu} \Delta_{\rho\sigma} \right] (x; x') \equiv \left\langle 0 \left| T \left\{ \psi_{\mu\nu}(x) \psi_{\rho\sigma}(x') \right\} \right| 0 \right\rangle \quad , \quad (3.35a)$$

$$i \Delta(x; x') \equiv \left\langle 0 \left| T \left\{ \phi(x) \phi(x') \right\} \right| 0 \right\rangle \quad , \quad (3.35b)$$

but we must consider the mixed propagator as well:

$$i \left[{}_{\mu\nu} \Delta \right] (x; x') \equiv \left\langle 0 \left| T \left\{ \psi_{\mu\nu}(x) \phi(x') \right\} \right| 0 \right\rangle \quad . \quad (3.35c)$$

Naturally, it is the propagators of the diagonal variables that we shall evaluate at first.

From (3.29) we conclude that:

$$\left\langle 0 \left| T \left\{ \zeta_{ij}(x) \zeta_{rs}(x') \right\} \right| 0 \right\rangle = i \Delta_A^s(x; x') \quad 2 \left[\delta_{i(r} \delta_{s)j} - \delta_{ij} \delta_{rs} \right] \quad , \quad (3.36a)$$

$$\left\langle 0 \left| T \left\{ \zeta_{0i}(x) \zeta_{0r}(x') \right\} \right| 0 \right\rangle = i \Delta_B^s(x; x') \quad \left[-\delta_{ir} \right] \quad , \quad (3.36b)$$

$$\left\langle 0 \left| T \left\{ v(x) v(x') \right\} \right| 0 \right\rangle = i \Delta_C^s(x; x') \quad , \quad (3.36c)$$

$$\left\langle 0 \left| T \left\{ \chi(x) \chi(x') \right\} \right| 0 \right\rangle = i \Delta_A^s(x; x') \quad , \quad (3.36d)$$

where the propagators $i \Delta_I^s(x; x')$ satisfy:

$$D_I^s i \Delta_I^s(x; x') = i \delta^{(4)}(x - x') \quad , \quad I = A, B, C \quad . \quad (3.37)$$

We transform back to the original fields with relations (3.22) and (3.25):

$$\left\langle 0 \left| T \left\{ \psi_{ij}(x) \psi_{rs}(x') \right\} \right| 0 \right\rangle = i \Delta_A^s(x; x') \quad 2 \left[\delta_{i(r} \delta_{s)j} - \delta_{ij} \delta_{rs} \right] + i \Delta_C^s(x; x') \quad \frac{s}{1+s} \delta_{ij} \delta_{rs}$$

$$+ i\Delta_A^s(x; x') \frac{1}{1+s} \delta_{ij} \delta_{rs} , \quad (3.38a)$$

$$\langle 0 | T \{ \psi_{0i}(x) \psi_{0r}(x') \} | 0 \rangle = i\Delta_B^s(x; x') [-\delta_{ir}] , \quad (3.38b)$$

$$\langle 0 | T \{ \psi_{00}(x) \psi_{00}(x') \} | 0 \rangle = i\Delta_A^s(x; x') \frac{1}{1+s} + i\Delta_C^s(x; x') \frac{s}{1+s} , \quad (3.38c)$$

$$\langle 0 | T \{ \psi_{00}(x) \psi_{rs}(x') \} | 0 \rangle = i\Delta_A^s(x; x') \frac{1}{1+s} \delta_{rs} + i\Delta_C^s(x; x') \frac{s}{1+s} \delta_{rs} , \quad (3.38d)$$

$$\langle 0 | T \{ \psi_{ij}(x) \phi(x') \} | 0 \rangle = -i\Delta_A^s(x; x') \frac{\sqrt{s}}{1+s} \delta_{ij} + i\Delta_C^s(x; x') \frac{\sqrt{s}}{1+s} \delta_{ij} , \quad (3.38e)$$

$$\langle 0 | T \{ \psi_{00}(x) \phi(x') \} | 0 \rangle = -i\Delta_A^s(x; x') \frac{\sqrt{s}}{1+s} + i\Delta_C^s(x; x') \frac{\sqrt{s}}{1+s} , \quad (3.38f)$$

$$\langle 0 | T \{ \phi(x) \phi(x') \} | 0 \rangle = i\Delta_A^s(x; x') \frac{s}{1+s} + i\Delta_C^s(x; x') \frac{1}{1+s} , \quad (3.38g)$$

To assemble expressions (3.38) in covariant form, we use (3.18) to derive an identity:

$$\delta_\mu^{(\alpha} \delta_\nu^{\beta)} = \bar{\delta}_\mu^{(\alpha} \bar{\delta}_\nu^{\beta)} + 2\delta_{(\mu}^0 \bar{\delta}_{\nu)}^{(\alpha} \delta_0^{\beta)} + \delta_\mu^0 \delta_\nu^0 \delta_0^\alpha \delta_0^\beta \quad \Longleftrightarrow \quad (3.39a)$$

$$\psi_{\mu\nu} = \psi_{\overline{\mu\nu}} + 2\delta_{(\mu}^0 \psi_{\overline{\nu)}} + \delta_\mu^0 \delta_\nu^0 \psi_{00} , \quad (3.39b)$$

which allows us to write the pseudo-graviton propagator as:

$$\begin{aligned} i[\mu\nu\Delta_{\rho\sigma}^s](x; x') &= i\Delta_A^s(x; x') \left\{ [\mu\nu T_{\rho\sigma}^A] + \frac{1}{1+s} [\mu\nu T_{\rho\sigma}^C] \right\} + i\Delta_B^s(x; x') [\mu\nu T_{\rho\sigma}^B] \\ &\quad + i\Delta_C^s(x; x') \frac{s}{1+s} [\mu\nu T_{\rho\sigma}^C] , \end{aligned} \quad (3.40)$$

where:

$$[\mu\nu T_{\rho\sigma}^A] \equiv 2[\bar{\eta}_{\mu(\rho} \bar{\eta}_{\sigma)\nu} - \bar{\eta}_{\mu\nu} \bar{\eta}_{\rho\sigma}] , \quad (3.41a)$$

$$[\mu\nu T_{\rho\sigma}^B] \equiv -4\delta_{(\mu}^0 \bar{\eta}_{\nu)(\rho} \delta_{\sigma)}^0 , \quad (3.41b)$$

$$[\mu\nu T_{\rho\sigma}^C] \equiv [\bar{\eta}_{\mu\nu} + \delta_\mu^0 \delta_\nu^0] [\bar{\eta}_{\rho\sigma} + \delta_\rho^0 \delta_\sigma^0] . \quad (3.41c)$$

There is also the mixed propagator which equals:

$$i[\mu\nu\Delta^s](x; x') = \left(-i\Delta_A^s + i\Delta_C^s \right) \frac{\sqrt{s}}{1+s} [\bar{\eta}_{\mu\nu} + \delta_\mu^0 \delta_\nu^0] . \quad (3.42)$$

3.4 The Explicit Form of the Propagators.

The condition (3.32) on the vacuum and the free field expansions (3.31) imply that:

$$i\Delta_I(x; x') = \int \frac{d^3k}{(2\pi)^3} \exp(-\epsilon k) \exp[i\vec{k} \cdot (\vec{x} - \vec{x}')] \left\{ \theta(\eta - \eta') \Psi(\eta, k, I) \Psi^*(\eta', k, I) \right. \\ \left. + \theta(\eta' - \eta) \Psi^*(\eta, k, I) \Psi(\eta', k, I) \right\} , \quad I = A, B, C . \quad (3.43a)$$

It is elementary to perform the angular integrations:

$$i\Delta_I(x; x') = \frac{1}{2\pi^2 \Delta x} \int_0^{+\infty} dk \exp(-\epsilon k) k \sin(k \Delta x) \left\{ \theta(\Delta \eta) \Psi(\eta, k, I) \Psi^*(\eta', k, I) \right. \\ \left. + \theta(-\Delta \eta) \Psi^*(\eta, k, I) \Psi(\eta', k, I) \right\} , \quad I = A, B, C , \quad (3.43b)$$

where $\Delta x \equiv \|\vec{x} - \vec{x}'\|$ and $\Delta \eta \equiv \eta - \eta'$. Notice that we have included the traditional ultraviolet convergence factor $\exp(-\epsilon k)$ which serves as an ultraviolet mode cutoff.

The actual computation of the propagators is a three-step process. First, we solve equation (3.34) to obtain the mode functions $\Psi(\eta, k, I)$. Then, we evaluate the propagators $i\Delta_I^s(x; x')$ by inserting the respective mode functions in (3.43). In the final step, we simply substitute the derived expressions for $i\Delta_I^s(x; x')$ in (3.38g) , (3.40) and (3.42) to arrive at the scalar, pseudo-graviton and mixed propagators respectively.

Step I. Consider, to begin with, the “A” modes equation:

$$D_A^s f(\eta, \vec{x}) = 0 \quad \implies \quad \Omega \left[-\partial_\eta^2 + \nabla^2 + \frac{2s^2 - s}{(1-s)^2} \frac{1}{\eta^2} \right] \Omega f(\eta, \vec{x}) = 0 . \quad (3.44)$$

Now rewrite (3.44) as follows:

$$\Omega^2 \left[-\partial_\eta^2 + \nabla^2 - \frac{2s}{1-s} \frac{1}{\eta} \partial_\eta \right] f(\eta, \vec{x}) = 0 , \quad (3.45)$$

and Fourier transform in momentum space:

$$\left[\partial_\eta^2 + \frac{2s}{1-s} \frac{1}{\eta} \partial_\eta + k^2 \right] \tilde{f}(\eta, \vec{k}) = 0 , \quad (3.46)$$

This can be cast in the form of the Bessel equation:

$$\left(\frac{d^2}{dy^2} + \frac{1}{y} \frac{d}{dy} + 1 - \frac{\nu^2}{y^2} \right) Z_\nu = 0 , \quad (3.47)$$

by substituting:

$$y \equiv k\eta \quad ; \quad g(y) \equiv y^w \, G(y) \equiv \tilde{f}(\eta, \vec{k}) \quad , \quad (3.48)$$

so that (3.46) becomes:

$$y^w \left[\frac{d^2}{dy^2} + \left(\frac{2s}{1-s} + 2w \right) \frac{1}{y} \frac{d}{dy} + 1 + w \left(\frac{2s}{1-s} + w - 1 \right) \frac{1}{y^2} \right] G(y) = 0 \quad . \quad (3.49)$$

This is the Bessel equation for $w = \frac{1-3s}{2(1-s)}$. The two solutions to (3.49) are Hankel functions:

$$G^{(1,2)}(y) = H_{\nu_A}^{(1,2)}(y) \quad , \quad \nu_A = \frac{1}{2} \left| \frac{1-3s}{1-s} \right| \quad , \quad (3.50)$$

and are complex conjugates of each other:

$$H_{\nu}^{(2)}(y) = H_{\nu}^{(1)*}(y) \quad . \quad (3.51)$$

A similar analysis is applied to the case of the “ B ” modes:

$$D_B^s f(\eta, \vec{x}) = 0 \quad \implies \quad \Omega \left[-\partial_\eta^2 + \nabla^2 + \frac{s}{(1-s)^2} \frac{1}{\eta^2} \right] \Omega f(\eta, \vec{x}) = 0 \quad . \quad (3.52)$$

In momentum space equation (3.52) becomes:

$$\left[\partial_\eta^2 + \frac{2s}{1-s} \frac{1}{\eta} \partial_\eta - \frac{2s}{1-s} \frac{1}{\eta^2} + k^2 \right] \tilde{f}(\eta, \vec{k}) = 0 \quad , \quad (3.53)$$

which is again equal – using (3.48) – to a Bessel equation when $w = \frac{1-3s}{2(1-s)}$:

$$y^w \left[\frac{d^2}{dy^2} + \left(\frac{2s}{1-s} + 2w \right) \frac{1}{y} \frac{d}{dy} + 1 + \left(w \left(\frac{2s}{1-s} + w - 1 \right) - \frac{2s}{1-s} \right) \frac{1}{y^2} \right] G(y) = 0 \quad . \quad (3.54)$$

The following set of complex conjugate solutions are implied:

$$G^{(1,2)}(y) = H_{\nu_B}^{(1,2)}(y) \quad , \quad \nu_B = \frac{1}{2} \left| \frac{1+s}{1-s} \right| \quad . \quad (3.55)$$

Finally, the “ C ” modes lead to the following equation:

$$\left[\partial_\eta^2 + \frac{2s}{1-s} \frac{1}{\eta} \partial_\eta - \frac{2(1+s)}{1-s} \frac{1}{\eta^2} + k^2 \right] \tilde{f}(\eta, \vec{k}) = 0 \quad . \quad (3.56)$$

The resulting Bessel equation is:

$$y^w \left[\frac{d^2}{dy^2} + \left(\frac{2s}{1-s} + 2w \right) \frac{1}{y} \frac{d}{dy} + 1 + \left(w \left(\frac{2s}{1-s} + w - 1 \right) - \frac{2(1+s)}{1-s} \right) \frac{1}{y^2} \right] G(y) = 0 \quad , \quad (3.57)$$

where $w = \frac{1-3s}{2(1-s)}$. The solutions are:

$$G^{(1,2)}(y) = H_{\nu_C}^{(1,2)}(y) \quad , \quad \nu_C = \frac{1}{2} \left| \frac{3-s}{1-s} \right| . \quad (3.58)$$

To obtain the expressions for the mode functions, we must identify the correspondence between Ψ and Ψ^* – as defined in (3.31) – and the solutions $H_\nu^{(1)}$ and $H_\nu^{(2)}$. This can be accomplished by requiring the normalizability of the vacuum. A useful asymptotic expansion of the two solutions is:

$$H_\nu^{(1)}(y) = \sqrt{\frac{2}{\pi y}} \exp \left[i \left(y - \frac{\pi}{2} \nu - \frac{\pi}{2} \right) \right] \left(1 + O(y^{-1}) \right) \quad , \quad y \gg 1 \quad , \quad (3.59a)$$

$$H_\nu^{(2)}(y) = \sqrt{\frac{2}{\pi y}} \exp \left[-i \left(y - \frac{\pi}{2} \nu - \frac{\pi}{2} \right) \right] \left(1 + O(y^{-1}) \right) \quad , \quad y \gg 1 \quad , \quad (3.59b)$$

Because of the exponential behaviour seen in (3.59) , we must associate $\Psi(\eta, k, I)$ – the coefficient function of the annihilation operator in (3.32) – with $H_{\nu_I}^{(2)}(k\eta)$ and $\Psi^*(\eta, k, I)$ with $H_{\nu_I}^{(1)}(k\eta)$. After we account for the normalization N and the transformation (3.48) , we get:

$$\Psi(\eta, k, I) = N(k) (k\eta)^w H_{\nu_I}^{(2)}(k\eta) \quad , \quad I = A, B, C \quad , \quad (3.60a)$$

$$\Psi^*(\eta, k, I) = N(k) (k\eta)^w H_{\nu_I}^{(1)}(k\eta) \quad , \quad I = A, B, C \quad . \quad (3.60b)$$

The alternate connection, namely $\Psi \leftrightarrow H^{(1)}$ and $\Psi^* \leftrightarrow H^{(2)}$, would lead to a vacuum state that is not normalizable. *

The normalization of the mode functions comes from (3.33). When we take into account relations (3.60) and the Hankel function identity:

$$H_\nu^{(2)}(y) \frac{d}{dy} H_\nu^{(1)}(y) - H_\nu^{(1)}(y) \frac{d}{dy} H_\nu^{(2)}(y) = \frac{4i}{\pi y} \quad , \quad (3.61)$$

* The normalizable vacuum state that follows from prescription (3.60) is the Bunch-Davies vacuum [4]. Although this works fine for power laws, there is an ambiguity for a generic scale factor. Since the solutions in the latter case are not known it is not clear whether a proper vacuum state exists in general.

equation (3.33) gives:

$$N(k) = \frac{1}{2}\sqrt{\pi} \eta_0^{\frac{s}{1-s}} k^{-\frac{1-3s}{2(1-s)}} , \quad (3.62)$$

so that:

$$\Psi(\eta, k, I) = \frac{1}{2}\sqrt{\pi\eta} \Omega^{-1}(\eta) H_{\nu_I}^{(2)}(k\eta) \quad , \quad I = A, B, C \quad , \quad (3.63a)$$

$$\Psi^*(\eta, k, I) = \frac{1}{2}\sqrt{\pi\eta} \Omega^{-1}(\eta) H_{\nu_I}^{(1)}(k\eta) \quad , \quad I = A, B, C \quad . \quad (3.63b)$$

The basic parameter values of the most physically interesting power laws are displayed in Table 1. The kinds of Hankel functions generated by inflation, matter, radiation and flatness are presented in Table 2 and their functional form in Table 3. The explicit expressions which the mode functions take in these cases are shown in Table 4. In the case of the inflating universe ($s = +\infty$) they are in agreement with previously obtained results [5].

	Flat	Radiation	Matter	Inflation
s	0	$\frac{1}{2}$	$\frac{2}{3}$	$+\infty$
$\Omega(\eta)$	1	$\frac{\eta}{\eta_0}$	$\frac{\eta^2}{\eta_0^2}$	$-\frac{\eta_0}{\eta}$

Table 1: The power law parameter s and scale factor $\Omega(\eta)$ for some special spacetimes.

$H_{\nu_I}^{(1,2)}(k\eta)$ Index	$s = 0$	$s = \frac{1}{2}$	$s = \frac{2}{3}$	$s = +\infty$
ν_A	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{3}{2}$
ν_B	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{5}{2}$	$\frac{1}{2}$
ν_C	$\frac{3}{2}$	$\frac{5}{2}$	$\frac{7}{2}$	$\frac{1}{2}$

Table 2: The value of the index of the Hankel functions for the special power law scale factors.

$H_\nu^{(2)}(k\eta)$ Index	Functional Form of $H_\nu^{(2)}(k\eta)$
$\nu = \frac{1}{2}$	$i\sqrt{\frac{2}{\pi k\eta}} \exp[-ik\eta]$
$\nu = \frac{3}{2}$	$-\sqrt{\frac{2}{\pi k\eta}} \exp[-ik\eta] \left(1 - \frac{i}{k\eta}\right)$
$\nu = \frac{5}{2}$	$-i\sqrt{\frac{2}{\pi k\eta}} \exp[-ik\eta] \left(1 - \frac{3i}{k\eta} - \frac{3}{(k\eta)^2}\right)$
$\nu = \frac{7}{2}$	$\sqrt{\frac{2}{\pi k\eta}} \exp[-ik\eta] \left(1 - \frac{6i}{k\eta} - \frac{15}{(k\eta)^2} + \frac{15i}{(k\eta)^3}\right)$

Table 3: The functional form of $H_\nu^{(2)}$ for the needed half-integer values of the index ν ; the form of $H_\nu^{(1)}$ is obtained by complex conjugation.

Power Law	Functional Form	Mode Function
$s = 0$	$\frac{i}{\sqrt{2k}} \exp[-ik\eta]$	$\Psi(\eta, k, A) = \Psi(\eta, k, B)$
$s = 0$	$\frac{i}{\sqrt{2k^3}} \frac{1}{\eta} (1 + ik\eta) \exp[-ik\eta]$	$\Psi(\eta, k, C)$
$s = \frac{1}{2}$	$\frac{i}{\sqrt{2k}} \frac{\eta_0}{\eta} \exp[-ik\eta]$	$\Psi(\eta, k, A)$
$s = \frac{1}{2}$	$\frac{i}{\sqrt{2k^3}} \frac{\eta_0}{\eta^2} (1 + ik\eta) \exp[-ik\eta]$	$\Psi(\eta, k, B)$
$s = \frac{1}{2}$	$\frac{i}{\sqrt{2k}} \frac{\eta_0}{\eta} \left(\frac{3}{(k\eta)^2} + \frac{3i}{k\eta} - 1\right) \exp[-ik\eta]$	$\Psi(\eta, k, C)$
$s = \frac{2}{3}$	$\frac{i}{\sqrt{2k^3}} \frac{\eta_0^2}{\eta^3} (1 + ik\eta) \exp[-ik\eta]$	$\Psi(\eta, k, A)$
$s = \frac{2}{3}$	$\frac{i}{\sqrt{2k}} \frac{\eta_0^2}{\eta^2} \left(\frac{3}{(k\eta)^2} + \frac{3i}{k\eta} - 1\right) \exp[-ik\eta]$	$\Psi(\eta, k, B)$
$s = \frac{2}{3}$	$\frac{1}{\sqrt{2k}} \frac{\eta_0^2}{\eta^2} \left(1 - \frac{6i}{k\eta} - \frac{15}{(k\eta)^2} + \frac{15i}{(k\eta)^3}\right) \exp[-ik\eta]$	$\Psi(\eta, k, C)$
$s = +\infty$	$-\frac{i}{\sqrt{2k^3}} \frac{1}{\eta_0} (1 + ik\eta) \exp[-ik\eta]$	$\Psi(\eta, k, A)$
$s = +\infty$	$-\frac{i}{\sqrt{2k}} \frac{\eta}{\eta_0} \exp[-ik\eta]$	$\Psi(\eta, k, B) = \Psi(\eta, k, C)$

Table 4: The form of $\Psi(\eta, k, I)$ for the special power law scale factors; the form of $\Psi^*(\eta, k, I)$ is obtained by complex conjugation.

Step II. We proceed to calculate the propagators $i\Delta_I^s(x; x')$ by substituting relations (3.63) in (3.43b) to obtain:

$$i\Delta_{\nu_I}(x; x') = \frac{\sqrt{\eta\eta'} \Omega^{-1}(\eta) \Omega^{-1}(\eta')}{8\pi\Delta x} \int_0^{+\infty} dk \exp(-\epsilon k) k \sin(k\Delta x) \left\{ \theta(\Delta\eta) H_{\nu_I}^{(2)}(k\eta) H_{\nu_I}^{(1)}(k\eta') + \theta(-\Delta\eta) H_{\nu_I}^{(1)}(k\eta) H_{\nu_I}^{(2)}(k\eta') \right\} . \quad (3.64)$$

An infrared divergence coming from the lower limit of integration may exist here. When present, it is due to the infinite size universe imposed by the integral approximation to the mode sums we have assumed. This divergence is not physical and has appeared in previous studies of the graviton propagator in de Sitter spacetime [6]. * On our finite spatial manifold the propagator is a mode sum over discrete momenta, so it is perfectly well defined. The integral approximation to this mode sum has lower limit $k_0 \sim R^{-1}$ [8].

It is interesting to examine the range of power laws for which this infrared divergence exists in the basic integral (3.64). The dominant behaviour of the Hankel functions for small k :

$$H_\nu^{(2)}(y) \quad \longmapsto \quad \frac{(\nu-1)! 2^\nu}{i\pi} y^{-\nu} \quad , \quad k \ll 1 \quad , \quad (3.65)$$

translates to the following behaviour for the propagators:

$$i\Delta_{\nu_I}(x; x') \quad \approx \quad \int_0^{+\infty} dk k^{2(1-\nu_I)} \quad , \quad k \ll 1 \quad , \quad (3.66)$$

and, therefore, the following condition for the presence of an infrared divergence:

$$1 - \nu_I \leq -\frac{1}{2} . \quad (3.67)$$

For the “A” modes (3.67) implies:

$$\nu_A = \frac{1}{2} \left| \frac{1-3s}{1-s} \right| \quad \implies \quad s \geq \frac{2}{3} \quad , \quad (3.68)$$

* For de Sitter spacetime, this does not preclude the appearance of infrared divergences in physical quantities [7,8].

while for the “B” and “C” modes we get:

$$\nu_B = \frac{1}{2} \left| \frac{1+s}{1-s} \right| \implies \frac{1}{2} \leq s \leq 2 , \quad (3.69a)$$

$$\nu_C = \frac{1}{2} \left| \frac{3-s}{1-s} \right| \implies 0 \leq s \leq \frac{3}{2} . \quad (3.69b)$$

Since the physical polarizations of the graviton are “A” modes, the corresponding crossover value $s_{\text{crit}} = \frac{2}{3}$ may have some indirect physical significance [9].

We now turn to the computation of the propagators $i\Delta_{\nu_I}(x; x')$ for the power laws of interest. As Table 2 indicates, four Hankel functions are needed. Substituting the expressions of Table 3 in (3.64), we get for $\nu = \frac{1}{2}$:

$$\begin{aligned} i\Delta_{\frac{1}{2}}(x; x') &= \frac{1}{4\pi^2 \Omega(\eta) \Omega(\eta') \Delta x} \int_0^{+\infty} dk \sin(k\Delta x) \exp[-ik(|\Delta\eta| - i\epsilon)] \\ &= \frac{1}{4\pi^2 \Omega(\eta) \Omega(\eta')} \frac{1}{(\Delta x - |\Delta\eta| + i\epsilon) (\Delta x + |\Delta\eta| - i\epsilon)} , \end{aligned} \quad (3.70)$$

for $\nu = \frac{3}{2}$:

$$\begin{aligned} i\Delta_{\frac{3}{2}}(x; x') &= \frac{1}{4\pi^2 \Omega(\eta) \Omega(\eta') \Delta x} \int_{k_0}^{+\infty} dk \sin(k\Delta x) \exp[-ik(|\Delta\eta| - i\epsilon)] \left[1 + \frac{1 + ik|\Delta\eta|}{k^2 \eta \eta'} \right] \\ &= i\Delta_{\frac{1}{2}}(x; x') - \frac{1}{8\pi^2 \Omega(\eta) \eta \Omega(\eta') \eta'} \left\{ -2 + \text{Ei}\left[ik_0(\Delta x - |\Delta\eta| + i\epsilon)\right] \right. \\ &\quad \left. + \text{Ei}\left[-ik_0(\Delta x + |\Delta\eta| - i\epsilon)\right] \right\} , \end{aligned} \quad (3.71)$$

for $\nu = \frac{5}{2}$:

$$\begin{aligned} i\Delta_{\frac{5}{2}}(x; x') &= \frac{1}{4\pi^2 \Omega(\eta) \Omega(\eta') \Delta x} \int_{k_0}^{+\infty} dk \sin(k\Delta x) \exp[-ik(|\Delta\eta| - i\epsilon)] \\ &\quad \left[1 + 3 \frac{1 + ik|\Delta\eta|}{k^2 \eta \eta'} + \frac{9 + 9ik|\Delta\eta| - 3k^2(\Delta\eta)^2}{k^4 \eta^2 \eta'^2} \right] \\ &= -2i\Delta_{\frac{1}{2}}(x; x') + 3i\Delta_{\frac{3}{2}}(x; x') \\ &\quad + \frac{1}{8\pi^2 \Omega(\eta) \eta^2 \Omega(\eta') \eta'^2} \left\{ 9k_0^{-2} - \frac{11}{2}(\Delta x)^2 + \frac{9}{2}(\Delta\eta)^2 + \frac{3}{2}\left[(\Delta x)^2 - (\Delta\eta)^2\right] \right. \\ &\quad \left. \times \left\{ \text{Ei}\left[ik_0(\Delta x - |\Delta\eta| + i\epsilon)\right] + \text{Ei}\left[-ik_0(\Delta x + |\Delta\eta| - i\epsilon)\right] \right\} \right\} , \end{aligned} \quad (3.72)$$

and for $\nu = \frac{7}{2}$:

$$\begin{aligned}
i\Delta_{\frac{7}{2}}(x; x') &= \frac{1}{4\pi^2 \Omega(\eta) \Omega(\eta') \Delta x} \int_{k_0}^{+\infty} dk \sin(k\Delta x) \exp\left[-ik(|\Delta\eta| - i\epsilon)\right] \\
&\quad \left[1 + 6 \frac{1 + ik|\Delta\eta|}{k^2 \eta \eta'} + \frac{45 + 45ik|\Delta\eta| - 15k^2(\Delta\eta)^2}{k^4 \eta^2 \eta'^2} \right. \\
&\quad \left. + \frac{225 + 225ik|\Delta\eta| - 90k^2\Delta\eta^2 - 15ik^3|\Delta\eta|^3}{k^6 \eta^3 \eta'^3} \right] \\
&= 5i\Delta_{\frac{1}{2}}(x; x') - 9i\Delta_{\frac{3}{2}}(x; x') + 5i\Delta_{\frac{5}{2}}(x; x') \\
&\quad + \frac{1}{16\pi^2 \Omega(\eta) \eta^3 \Omega(\eta') \eta'^3} \left\{ 225k_0^{-4} - \left[75(\Delta x)^2 - 45(\Delta\eta)^2 \right] k_0^{-2} \right. \\
&\quad + \left[\frac{137}{8}(\Delta x)^4 - \frac{125}{4}(\Delta x)^2(\Delta\eta)^2 + \frac{105}{8}(\Delta\eta)^4 \right] - \frac{15}{4} \left((\Delta x)^2 - (\Delta\eta)^2 \right)^2 \\
&\quad \left. \times \left\{ \text{Ei}\left[ik_0(\Delta x - |\Delta\eta| + i\epsilon) \right] + \text{Ei}\left[-ik_0(\Delta x + |\Delta\eta| - i\epsilon) \right] \right\} \right\} , \quad (3.73)
\end{aligned}$$

The integration technique consists of performing as many integrations by parts as needed to reduce the original expression to the following basic integrals:

$$\int_0^{+\infty} dk \sin(k\Delta x) \exp\left[-ik(|\Delta\eta| - i\epsilon)\right] = \frac{\Delta x}{(\Delta x - |\Delta\eta| + i\epsilon)(\Delta x + |\Delta\eta| - i\epsilon)} , \quad (3.74)$$

$$\begin{aligned}
\int_{k_0}^{+\infty} \frac{dk}{k} \cos(k\Delta x) \exp\left[-ik(|\Delta\eta| - i\epsilon)\right] &= -\frac{1}{2} \left\{ \text{Ei}\left[ik_0(\Delta x - |\Delta\eta| + i\epsilon) \right] \right. \\
&\quad \left. + \text{Ei}\left[-ik_0(\Delta x + |\Delta\eta| - i\epsilon) \right] \right\} . \quad (3.75)
\end{aligned}$$

The first of these does not exhibit an infrared divergence but the second does and, therefore, its lower limit of integration is regulated to the value $k_0 \sim R^{-1}$. The exponential integral function $\text{Ei}(x)$ is defined as [10]:

$$\text{Ei}(-x) = - \int_x^{+\infty} dt t^{-1} \exp(-t) , \quad (3.76)$$

and for small x is very well approximated by:

$$\text{Ei}(-x) \approx \gamma + \ln x , \quad x \ll 1 , \quad (3.77)$$

where γ stands for Euler's constant. It is legitimate to employ this approximation in our case:

$$\int_{k_0}^{+\infty} \frac{dk}{k} \cos(k\Delta x) \exp\left[-ik(|\Delta\eta| - i\epsilon)\right] \approx$$

$$- \frac{1}{2} \left\{ \ln \left[k_0^2 (\Delta x - |\Delta \eta| + i\epsilon) (\Delta x + |\Delta \eta| - i\epsilon) \right] + 2\gamma \right\} , \quad (3.78)$$

since we have assumed throughout that $k_0 \Delta x \ll 1$ and $k_0 |\Delta \eta| \ll 1$.

Using the approximation (3.78) and the definition:

$$(x - x')^2 \equiv (\Delta x - |\Delta \eta| + i\epsilon) (\Delta x + |\Delta \eta| - i\epsilon) , \quad (3.79)$$

we can express $i\Delta_{\nu_I}(x; x')$ in a more economical form:

$$i\Delta_{\frac{1}{2}}(x; x') = \frac{1}{4\pi^2 \Omega(\eta) \Omega(\eta')} \frac{1}{(x - x')^2} , \quad (3.80)$$

$$i\Delta_{\frac{3}{2}}(x; x') = \frac{1}{8\pi^2 \Omega(\eta) \eta \Omega(\eta') \eta'} \left\{ \frac{2\eta \eta'}{(x - x')^2} - \ln \left[k_0^2 (x - x')^2 \right] - 2(\gamma - 1) \right\} , \quad (3.81)$$

$$i\Delta_{\frac{5}{2}}(x; x') = \frac{1}{8\pi^2 \Omega(\eta) \eta^2 \Omega(\eta') \eta'^2} \left\{ \frac{2\eta^2 \eta'^2}{(x - x')^2} + \frac{3}{2} \left[(x - x')^2 - 2\eta \eta' \right] \ln \left[k_0^2 (x - x')^2 \right] \right. \\ \left. + 9k_0^{-2} - \frac{11}{2}(\Delta x)^2 + \frac{9}{2}(\Delta \eta)^2 + 6\eta \eta' + 3\gamma \left[(x - x')^2 - 2\eta \eta' \right] \right\} , \quad (3.82)$$

$$i\Delta_{\frac{7}{2}}(x; x') = \frac{1}{16\pi^2 \Omega(\eta) \eta^3 \Omega(\eta') \eta'^3} \left\{ \frac{4\eta^3 \eta'^3}{(x - x')^2} - \left[\frac{15}{4}(x - x')^4 - 5(x - x')^2 \eta \eta' + 12\eta^2 \eta'^2 \right] \right. \\ \times \left(\ln \left[k_0^2 (x - x')^2 \right] + 2\gamma \right) + 225k_0^{-4} - \left[75(\Delta x)^2 - 45(\Delta \eta)^2 - 90\eta \eta' \right] k_0^{-2} \\ + \left[\frac{137}{8}(\Delta x)^4 - \frac{125}{4}(\Delta x)^2 (\Delta \eta)^2 + \frac{105}{8}(\Delta \eta)^4 \right] \\ \left. - \left[55(\Delta x)^2 - 45(\Delta \eta)^2 \right] \eta \eta' + 24\eta^2 \eta'^2 \right\} . \quad (3.83)$$

Step III. To obtain the explicit form of the scalar, pseudo-graviton and mixed propagators for the power laws of Table 1, we only need to appropriately substitute the above expressions in equations (3.38g), (3.40) and (3.42) respectively. As an example we shall consider the case of de Sitter spacetime so that we can connect with previous work [3,8]. It corresponds to $s \rightarrow +\infty$ and in this limit the scalar field action becomes the cosmological constant term:

$$\dot{\varphi}^2 \rightarrow 0 \quad ; \quad V(\varphi) \rightarrow (8\pi G) \Lambda^{-1} , \quad (3.84)$$

where $\Lambda = 3\eta_0^{-2}$. Consequently, only the pseudo-graviton propagator exists and it equals:

$$\begin{aligned}
i \left[\mu\nu \Delta_{\rho\sigma} \right]_{s=+\infty} (x; x') &= i\Delta_{\frac{3}{2}}(x; x') \left[\mu\nu T_{\rho\sigma}^A \right] + i\Delta_{\frac{1}{2}}(x; x') \left\{ \left[\mu\nu T_{\rho\sigma}^B \right] + \left[\mu\nu T_{\rho\sigma}^C \right] \right\} \\
&= i\Delta_{\frac{1}{2}}(x; x') \left\{ \left[\mu\nu T_{\rho\sigma}^A \right] + \left[\mu\nu T_{\rho\sigma}^B \right] + \left[\mu\nu T_{\rho\sigma}^C \right] \right\} \\
&\quad - \frac{1}{8\pi^2\eta_0^2} \left\{ \ln \left[k_0^2 (x - x')^2 \right] + 2(\gamma - 1) \right\} \left[\mu\nu T_{\rho\sigma}^A \right] \\
&= \frac{1}{8\pi^2\eta_0^2} \left\{ \frac{2\eta \eta'}{(x - x')^2} \left(\left[\mu\nu T_{\rho\sigma}^A \right] + \left[\mu\nu T_{\rho\sigma}^B \right] + \left[\mu\nu T_{\rho\sigma}^C \right] \right) \right. \\
&\quad \left. - \left\{ \ln \left[k_0^2 (x - x')^2 \right] + 2(\gamma - 1) \right\} \left[\mu\nu T_{\rho\sigma}^A \right] \right\} , \quad (3.85)
\end{aligned}$$

in agreement with previous results. By using the zero modes present we can absorb the constant term $2(\gamma - 1)$ and set $k_0 = \eta_0^{-1}$ [8]. However, this property is particular to the de Sitter spacetime and is not shared by the other power laws of interest. Finally, the combination of the tensor factors (3.41) that appears in (3.85) equals:

$$\left[\mu\nu T_{\rho\sigma}^A \right] + \left[\mu\nu T_{\rho\sigma}^B \right] + \left[\mu\nu T_{\rho\sigma}^C \right] = 2\eta_{\mu(\rho} \eta_{\sigma)\nu} - \eta_{\mu\nu} \eta_{\rho\sigma} . \quad (3.86)$$

3.5 The Ghost Sector.

The ghost Lagrangian is obtained by varying the gauge functional F_μ given by (3.14b).

Under an infinitesimal coordinate change:

$$y'^\mu = y^\mu + \kappa \omega^\mu(y) , \quad (3.87)$$

the full metric transforms to:

$$g'_{\mu\nu}(x) = \frac{\partial y^\rho}{\partial y'^\mu}(x) \frac{\partial y^\sigma}{\partial y'^\nu}(x) g_{\rho\sigma}(y'^{-1}(x)) , \quad (3.88)$$

so that the infinitesimal variation of the pseudo-graviton field is:

$$\delta\psi_{\mu\nu} = \delta_0\psi_{\mu\nu} + \kappa \delta_1\psi_{\mu\nu} , \quad (3.89a)$$

$$\delta_0\psi_{\mu\nu} = - \left[2\omega_{(\mu,\nu)} + 2\eta_{\mu\nu} \omega^\rho (\ln \Omega)_{,\rho} \right] , \quad (3.89b)$$

$$\delta_1\psi_{\mu\nu} = - \left[2\omega^\rho_{,(\mu} \psi_{\nu)\rho} + \omega^\rho \psi_{\mu\nu,\rho} + 2\omega^\rho \psi_{\mu\nu} (\ln \Omega)_{,\rho} \right] , \quad (3.89c)$$

where we have decomposed $\delta\psi_{\mu\nu}$ into a term with no $\psi_{\mu\nu}$ and a term linear in $\psi_{\mu\nu}$. There is a similar transformation on the full scalar field:

$$\varphi'(x) = \varphi\left(y'^{-1}(x)\right) , \quad (3.90)$$

and a corresponding infinitesimal variation on the fluctuating scalar field:

$$\delta\phi = \delta_0\phi + \delta_1\phi , \quad (3.91a)$$

$$\delta_0\phi = -\kappa\omega^0 \widehat{\phi}' \quad ; \quad \delta_1\phi = -\kappa\omega^\mu \phi_{,\mu} , \quad (3.91b)$$

The ghost Lagrangian is taken to be:

$$\mathcal{L}_{\text{gh}} = -\Omega \bar{\omega}^\mu \delta F_\mu . \quad (3.92)$$

As usual, the infinitesimal parameter of the coordinate transformation (3.87) is the anticommuting ghost field ω and $\bar{\omega}$ is the antighost field. Up to a surface term the ghost interaction Lagrangian is:

$$\mathcal{L}_{\text{gh}}^{(3)} = -\Omega \bar{\omega}^\mu \delta_1 F_\mu \quad (3.93a)$$

$$\begin{aligned} &= -\kappa\Omega^2 \bar{\omega}^{\mu,\nu} \left[\psi_{\mu\rho} \omega^{\rho}_{,\nu} + \psi_{\nu\rho} \omega^{\rho}_{,\mu} + \psi_{\mu\nu,\rho} \omega^\rho + 2\psi_{\mu\nu} \omega^\rho (\ln \Omega)_{,\rho} \right] \\ &\quad + \kappa \left(\Omega^2 \bar{\omega}^\rho \right)_{,\rho} \left[\psi_{\mu\nu} \omega^{\mu,\nu} + \frac{1}{2} \psi_{,\nu} \omega^\nu + \psi \omega^\nu (\ln \Omega)_{,\nu} \right] \\ &\quad + \kappa^2 \Omega^2 \widehat{\phi}' \bar{\omega}^\mu \eta_{\mu 0} \phi_{,\rho} \omega^\rho , \end{aligned} \quad (3.93b)$$

and completely determines the ghost-graviton and ghost-scalar interactions.

The quadratic part of the ghost action is:

$$\mathcal{L}_{\text{gh}}^{(2)} = -\Omega \bar{\omega}^\mu \delta_0 F_\mu \quad (3.94a)$$

$$= \bar{\omega}^\mu \left[D_A \bar{\delta}_\mu^\rho + D_B \delta_\mu^0 \delta_0^\rho \right] \omega_\rho , \quad (3.94b)$$

leading to the following ghost propagator:

$$i \left[{}_\mu \Delta_\rho \right] (x, x') = i \Delta_A(x, x') \bar{\eta}_{\mu\rho} - i \Delta_B(x, x') \eta_{\mu 0} \eta_{\rho 0} . \quad (3.95)$$

It is not possible to explicitly compute the ghost propagator for a generic scale factor. However, in the case of the power law scale factors of Table 1 we get:

$$\begin{aligned} i \left[\mu \Delta_\rho \right]_{s=0} (x; x') &= i \Delta_{\frac{1}{2}} (x, x') \left[\bar{\eta}_{\mu\rho} - \eta_{\mu 0} \eta_{\rho 0} \right] \\ &= \frac{1}{4\pi^2} \frac{1}{(x - x')^2} \eta_{\mu\rho} \ , \end{aligned} \quad (3.96)$$

$$\begin{aligned} i \left[\mu \Delta_\rho \right]_{s=\frac{1}{2}} (x; x') &= i \Delta_{\frac{1}{2}} (x, x') \bar{\eta}_{\mu\rho} - i \Delta_{\frac{3}{2}} (x, x') \eta_{\mu 0} \eta_{\rho 0} \\ &= \frac{\eta_0^2}{8\pi^2 \eta^2 \eta'^2} \left\{ \frac{2\eta \eta'}{(x - x')^2} \eta_{\mu\rho} + \left(\ln \left[k_0^2 (x - x')^2 \right] \right. \right. \\ &\quad \left. \left. + 2(\gamma - 1) \right) \eta_{\mu 0} \eta_{\rho 0} \right\} \ , \end{aligned} \quad (3.97)$$

$$\begin{aligned} i \left[\mu \Delta_\rho \right]_{s=\frac{2}{3}} (x; x') &= i \Delta_{\frac{3}{2}} (x, x') \bar{\eta}_{\mu\rho} - i \Delta_{\frac{5}{2}} (x, x') \eta_{\mu 0} \eta_{\rho 0} \\ &= \frac{\eta_0^4}{8\pi^2 \eta^3 \eta'^3} \left\{ \frac{2\eta \eta'}{(x - x')^2} - \left(\ln \left[k_0^2 (x - x')^2 \right] + 2(\gamma - 1) \right) \right\} \eta_{\mu\rho} \\ &\quad - \frac{\eta_0^4}{8\pi^2 \eta^4 \eta'^4} \left\{ \left[\frac{3}{2} (x - x')^2 - 2\eta \eta' \right] \left(\ln \left[k_0^2 (x - x')^2 \right] + 2\gamma \right) \right. \\ &\quad \left. + 4\eta\eta' + \frac{9}{k_0^2} - \frac{11}{2} (\Delta x)^2 + \frac{9}{2} (\Delta \eta)^2 \right\} \eta_{\mu 0} \eta_{\nu 0} \ . \end{aligned} \quad (3.98)$$

$$\begin{aligned} i \left[\mu \Delta_\rho \right]_{s=+\infty} (x; x') &= i \Delta_{\frac{3}{2}} (x, x') \bar{\eta}_{\mu\rho} - i \Delta_{\frac{1}{2}} (x, x') \eta_{\mu 0} \eta_{\rho 0} \\ &= \frac{1}{8\pi^2 \eta_0^2} \left\{ \frac{2\eta \eta'}{(x - x')^2} \eta_{\mu\rho} - \left(\ln \left[k_0^2 (x - x')^2 \right] \right. \right. \\ &\quad \left. \left. + 2(\gamma - 1) \right) \bar{\eta}_{\mu\rho} \right\} \ , \end{aligned} \quad (3.99)$$

Again, when $s = +\infty$, there is agreement with the previously obtained result in de Sitter spacetime [3].

4. Corrections to the Newtonian gravitational force

An elementary application of the perturbative tools developed above is the computation of the linearized response of the gravitational field to a point source. We may as well choose a matter dominated universe so that in the appropriate limit we can deduce the gravitational force law and compare it with the Newtonian law we currently measure.

The gravitational force is obtained from the geodesic equation the worldline $\chi^\mu(\tau)$ of a particle obeys:

$$\ddot{\chi}^\mu(\tau) + \Gamma_{\rho\sigma}^\mu[\chi(\tau)] \dot{\chi}^\rho(\tau) \dot{\chi}^\sigma(\tau) = 0 \quad . \quad (4.1)$$

Any deviation of its worldline from the geodesic trajectory it would occupy in the absence of gravitational sources will be attributed to a gravitational force. To make contact with the Newtonian theory we shall consider the slow motion of the particle in a weak stationary gravitational field. In that case, the geodesic equation simplifies considerably:

$$\ddot{\chi}_i = \frac{\kappa}{2\Omega^4} h_{00,i} \quad , \quad (4.2)$$

where we have chosen $\dot{\chi}^0(\tau) = \Omega^{-1}(\chi^0)$. The gravitational force is that part of the physical acceleration which is not attributable to the Hubble flow. Hence we can write:

$$F_i = m \Omega \ddot{\chi}_i \quad , \quad (4.3)$$

which, with (4.2), gives:

$$F_i = \frac{\kappa m}{2\Omega^3} h_{00,i} \quad . \quad (4.4)$$

The weak stationary gravitational field will be assumed as due to the presence of a point source. The action of the source worldline $\chi_s^\mu(\tau)$ is:

$$\mathcal{S}_s = -m_s \int d\tau \sqrt{-g_{\mu\nu}[\chi_s(\tau)] \dot{\chi}_s^\mu(\tau) \dot{\chi}_s^\nu(\tau)} \quad , \quad (4.5)$$

and the resulting stress tensor:

$$T_s^{\mu\nu}(x) = \frac{m_s}{\sqrt{-g(x)}} \int d\tau \frac{\dot{\chi}_s^\mu(\tau) \dot{\chi}_s^\nu(\tau) \delta^{(4)}(x - \chi_s(\tau))}{\sqrt{-g_{\rho\sigma}[\chi_s(\tau)] \dot{\chi}_s^\rho(\tau) \dot{\chi}_s^\sigma(\tau)}} \quad . \quad (4.6)$$

The stationary nature of the source allows us to choose:

$$\dot{\chi}_s^\mu(\tau) = \Omega^{-1}(\chi^0) \delta_0^\mu , \quad (4.7)$$

in which case the stress tensor simplifies to:

$$T_s^{\mu\nu}(\eta, \vec{x}) = m_s \Omega^{-5}(\eta) \delta_0^\mu \delta_0^\nu \delta^{(3)}(\vec{x}) . \quad (4.8)$$

The linearized response of the gravitational field to the presence of the point source is given by [3]:

$$\psi_{\mu\nu}(\eta, \vec{x}) = -\frac{\kappa}{2} \int d\eta' \int d^3x' \Omega^6(\eta') \left[{}_{\mu\nu}G^{\text{ret}}_{\rho\sigma} \right](x; x') T_s^{\rho\sigma}(x') , \quad (4.9)$$

where the pseudo-graviton retarded Green's function is defined as:

$$\left[{}_{\mu\nu}G^{\text{ret}}_{\rho\sigma} \right](x; x') \equiv 2\theta(\Delta\eta) \text{Im} \left[i \left[{}_{\mu\nu}\Delta_{\rho\sigma} \right](x; x') \right] . \quad (4.10)$$

For the power law scale factors we have analyzed, the retarded propagators $G_I^{\text{ret}}(x; x')$ can be expressed as follows:

$$G_I^{\text{ret}}(x; x') = 2\theta(\Delta\eta) \text{Im} \left[i \Delta_I^s(x; x') \right] . \quad (4.11)$$

In terms of the mode functions:

$$G_I^{\text{ret}}(x; x') = \frac{\theta(\Delta\eta)}{\pi^2 \Delta x} \int_0^{+\infty} dk \, k \sin(k\Delta x) \text{Im} \left[\Psi(\eta, k, I) \Psi^*(\eta', k, I) \right] , \quad (4.12)$$

where we have used (3.43b).

Inspection of the form of the three tensor factors (3.41) and the stress tensor (4.8) shows that only the $\delta_\mu^0 \delta_\nu^0 \delta_\rho^0 \delta_\sigma^0$ part of $\left[{}_{\mu\nu}T^C_{\rho\sigma} \right]$ can contribute to the desired $\kappa\psi_{00}(\eta, \vec{x})$ response.

$$\psi_{00}(\eta, \vec{x}) = -\frac{\kappa}{2} \int d\eta' \int d^3x' \Omega^6(\eta') \left[\frac{1}{1+s} G_A^{\text{ret}}(x; x') + \frac{s}{1+s} G_C^{\text{ret}}(x; x') \right] \delta_\rho^0 \delta_\sigma^0 T_s^{\rho\sigma}(x') . \quad (4.13)$$

Specializing to the matter dominated universe leads to:

$$G_A^{\text{ret}} \Big|_{s=\frac{2}{3}}(x; x') = \frac{\theta(\Delta\eta)}{2\pi^2 \Delta x} \frac{\eta_0^4}{\eta^3 \eta'^3} \int_0^{+\infty} dk \sin(k\Delta x) \left\{ \cos(k\Delta\eta) \left[\frac{\Delta\eta}{k} \right] + \sin(k\Delta\eta) \left[-\frac{1}{k^2} - \eta \eta' \right] \right\} , \quad (4.14a)$$

$$G_C^{\text{ret}} \Big|_{s=\frac{2}{3}}(x; x') = \frac{\theta(\Delta\eta)}{2\pi^2 \Delta x} \frac{\eta_0^4}{\eta^5 \eta'^5} \int_0^{+\infty} dk \sin(k\Delta x) \times \left\{ \cos(k\Delta\eta) \left[\frac{225\Delta\eta}{k^5} - \frac{15(\Delta\eta)^3}{k^3} + \frac{45\eta \eta' \Delta\eta}{k^3} + \frac{6\eta^2 \eta'^2 \Delta\eta}{k} \right] + \sin(k\Delta\eta) \left[-\frac{225}{k^6} + \frac{90(\Delta\eta)^2}{k^4} - \frac{45\eta \eta'}{k^4} + \frac{15\eta \eta' (\Delta\eta)^2}{k^2} - \frac{6\eta^2 \eta'^2}{k^2} - \eta^3 \eta'^3 \right] \right\} , \quad (4.14b)$$

where we have used Table 4 to substitute the appropriate functional form of the mode function and its complex conjugate in (4.12). We then perform all integrations by parts needed to express (4.14) in terms of the following integrals:

$$I_{\sin} \equiv \int_0^{+\infty} dk \sin(k\Delta x) \sin(k\Delta\eta) = \frac{\pi}{2} \delta(\Delta x - \Delta\eta) - \frac{\pi}{2} \delta(\Delta x + \Delta\eta) , \quad (4.15a)$$

$$I_{\cos} \equiv \int_0^{+\infty} \frac{dk}{k} \cos(k\Delta x) \sin(k\Delta\eta) = \frac{\pi}{2} \theta(\Delta x + \Delta\eta) - \frac{\pi}{2} \theta(\Delta x - \Delta\eta) . \quad (4.15b)$$

The result is:

$$G_A^{\text{ret}} \Big|_{s=\frac{2}{3}}(x; x') = -\frac{\theta(\Delta\eta)}{2\pi^2} \frac{\eta_0^4}{\eta^2 \eta'^2} \left\{ \frac{1}{\Delta x} I_{\sin} + \frac{1}{\eta \eta'} I_{\cos} \right\} , \quad (4.16a)$$

$$G_C^{\text{ret}} \Big|_{s=\frac{2}{3}}(x; x') = -\frac{\theta(\Delta\eta)}{2\pi^2} \frac{\eta_0^4}{\eta^2 \eta'^2} \left\{ \frac{1}{\Delta x} I_{\sin} + \left[\frac{6}{\eta \eta'} - \frac{15(x-x')^2}{2\eta^2 \eta'^2} + \frac{15(x-x')^4}{8\eta^3 \eta'^3} \right] I_{\cos} \right\} , \quad (4.16b)$$

which finally implies: *

$$G_A^{\text{ret}} \Big|_{s=\frac{2}{3}}(x; x') = -\frac{\theta(\Delta\eta)}{4\pi} \frac{\eta_0^4}{\eta^2 \eta'^2} \left\{ \frac{1}{\Delta x} \delta(\Delta x - \Delta\eta) + \frac{1}{\eta \eta'} \theta(\Delta\eta - \Delta x) \right\} , \quad (4.17a)$$

$$G_C^{\text{ret}} \Big|_{s=\frac{2}{3}}(x; x') = -\frac{\theta(\Delta\eta)}{4\pi} \frac{\eta_0^4}{\eta^2 \eta'^2} \left\{ \frac{1}{\Delta x} \delta(\Delta x - \Delta\eta) + \left[\frac{6}{\eta \eta'} - \frac{15(x-x')^2}{2\eta^2 \eta'^2} + \frac{15(x-x')^4}{8\eta^3 \eta'^3} \right] \theta(\Delta\eta - \Delta x) \right\} . \quad (4.17b)$$

* These retarded Green's functions could also be obtained directly from the imaginary part of the $i\Delta_{\frac{7}{2}}(x; x')$ propagator (3.83).

When we insert (4.8) and (4.17) in equation (4.13) and use the delta function present in $T_s^{\rho\sigma}(x')$ to do all three spatial integrations, we get:

$$\psi_{00}(\eta, \vec{x}) = \frac{\kappa m_s}{8\pi} \frac{\eta_0^2}{\eta^2} \left\{ \frac{1}{x} + \int_{\eta_1}^{\eta-x} d\eta' \left[\frac{3(x^2 - \eta^2)^2}{4\eta^3 \eta'^3} - \frac{3(x^2 - \eta^2)}{2\eta^3 \eta'} + \frac{3\eta'}{4\eta^3} \right] \right\} , \quad (4.18)$$

where $x = \|\vec{x}\|$ and η_1 is the beginning of matter domination in the universe. In terms of the graviton field $h_{\mu\nu} = \Omega^2 \psi_{\mu\nu}$ the conformal time integration gives:

$$h_{00}(\eta, \vec{x}) = \frac{\kappa m_s}{16\pi} \Omega \left\{ \frac{2}{x} - \frac{3\eta_1^2}{4\eta^3} - \frac{3x}{\eta^2} - \frac{3(x^2 - \eta^2)}{\eta^3} \ln\left(\frac{\eta - x}{\eta_1}\right) + \frac{3(x^2 - \eta^2)^2}{4\eta^3 \eta_1^2} \right\} . \quad (4.19)$$

The gravitational force is obtained from (4.4):

$$F(\eta, r) = -G m m_s \left\{ \frac{1}{r^2} + \frac{3}{\Omega^2 \eta^2} + \frac{3r}{2\Omega^3 \eta^3} + \frac{3r}{2\Omega^3 \eta \eta_1^2} + \frac{3r}{\Omega^3 \eta^3} \ln\left(\frac{\Omega\eta - r}{\Omega\eta_1}\right) - \frac{3r^3}{2\Omega^5 \eta^3 \eta_1^2} \right\} , \quad (4.20)$$

and has been expressed in terms of the physical distance $\vec{r} = \Omega \vec{x}$. The first term in (4.20) is the Newtonian attractive inverse square law. The remaining terms – one of which is independent of the distance – represent the corrections. To get an estimate for their relative strength, we apply (1.3)-(1.4) to a matter dominated universe:

$$\eta = 3 t_0^{\frac{2}{3}} t^{\frac{1}{3}} \quad ; \quad \eta_0 = 3 t_0 \quad , \quad (4.21)$$

and re-express the force as:

$$F(t, r) = -G m m_s \left\{ \frac{1}{r^2} + \frac{1}{3t^2} + \frac{r}{18t^3} + \frac{1}{18} r t^{-\frac{7}{3}} t_1^{-\frac{2}{3}} + \frac{r}{9t^3} \ln\left[\left(t - \frac{1}{3}r\right) t^{-\frac{2}{3}} t_1^{-\frac{1}{3}}\right] - \frac{1}{162} r^3 t^{-\frac{13}{3}} t_1^{-\frac{2}{3}} \right\} , \quad (4.22)$$

The beginning of matter domination is thought to have occurred at a physical time t_1 such that [11]:

$$\frac{a(t)}{a(t_1)} \approx 10^4 \quad \implies \quad t = 10^6 t_1 \quad , \quad (4.23)$$

where, of course, t is the present. Using (4.23) we obtain:

$$F(t, r) = -\frac{G m m_s}{r^2} \left\{ 1 + \frac{1}{3} \left(\frac{r}{t}\right)^2 + \frac{5000}{9} \left(\frac{r}{t}\right)^3 + \frac{1}{9} \left(\frac{r}{t}\right)^3 \ln\left[\frac{100}{3}\left(3-\frac{r}{t}\right)\right] - \frac{5000}{81} \left(\frac{r}{t}\right)^5 \right\} \quad (4.24)$$

It is now apparent that the correction terms are negligible unless we consider objects of size comparable to that of the observed universe. The present time t is of the order of 10^{28} cm while a galaxy and a cluster of galaxies have typical size $r \approx 10^{23}$ cm and $r \approx 10^{25}$ cm respectively.

5. Epilogue

One possible explanation of the discrepancy between the predictions of the Newtonian theory and the data provided by the galactic rotation curves, is an appropriate modification of the Newtonian gravitational force law at large distances. A natural way for this to occur would be due to the deviations from flat space introduced by the matter dominated background. These deviations turn out to be unable to bridge the discrepancy by themselves or to, at least, reduce significantly the large amount of dark matter required for that purpose.

In retrospect, it would have been very hard for such corrections to have the desired effect. For if they were strong at galactic scales to resolve the rotation curves mystery, they would become enormous at even larger scales something which is ruled out by the existence of structures much larger than galaxies: there is no gravitational confinement at galactic scales. The corrections would have to be of the proper strength at galactic scales and weak at all other scales.

From the quantum field theoretic point of view, the Feynman rules were developed for a spatially flat Robertson-Walker background generated by a dynamical scalar source. This is a much wider class of backgrounds than what is needed for the currently observed

universe. Their study revealed that conformal flatness is a more powerful organizing principle than maximal symmetry. This fact becomes especially obvious from the identity of the expressions for the quadratic Lagrangian in the case of de Sitter spacetime (maximally symmetric) and flat Robertson-Walker spacetimes (not maximally symmetric).

Moreover, the Feynman rules obtained make possible any perturbative calculation desired for power law scale factors. For instance, the existence of strong infrared quantum gravitational effects could be studied in detail in these backgrounds.

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